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# Approximate perfect differential equations of second order

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## Abstract

In this paper we prove the Hyers-Ulam stability of the perfect linear differential equation  $f(t)y''(t) + f_1(t)y'(t) + f_2(t)y(t) = Q(t)$ , where  $f, y \in C^2[a, b]$ ,  $Q \in C[a, b]$ ,  $f_2(t) = f_1'(t) - f''(t)$  and  $-\infty < a < b < +\infty$ .

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**Keywords:** Hyers-Ulam stability; differential equation

## 1 Introduction

The question concerning the stability of group homomorphisms was posed by Ulam [1]. Hyers [2] solved the case of approximately additive mappings in Banach spaces and T.M. Rassias generalized the result of Hyers [3].

**Definition 1.1** Let  $X$  be a normed space over a scalar field  $\mathbb{K}$  and let  $I$  be an open interval. Assume that  $a_0, a_1, \dots, a_n, h : I \rightarrow \mathbb{K}$  are continuous functions. We say that the differential equation

$$a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \dots + a_1(t)y'(t) + a_0(t)y(t) + h(t) = 0 \quad (1.1)$$

has the Hyers-Ulam stability if, for any function  $f : I \rightarrow X$  satisfying the differential inequality

$$\|a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \dots + a_1(t)y'(t) + a_0(t)y(t) + h(t)\| \leq \varepsilon$$

for all  $t \in I$  and some  $\varepsilon \geq 0$ , there exists a solution  $g : I \rightarrow X$  of (1.1) such that  $\|f(t) - g(t)\| \leq K(\varepsilon)$  for all  $t \in I$ , where  $K(\varepsilon)$  is a function depending only on  $\varepsilon$ .

Obłozza [4, 5] was the first author who investigated the Hyers-Ulam stability of differential equations (also see [6]).

Jung [7] solved the inhomogeneous differential equation of the form  $y'' + 2xy' - 2ny = \sum_{m=0}^{\infty} a_m x^m$ , where  $n$  is a positive integer, and he used this result to prove the Hyers-Ulam stability of the differential equation  $y'' + 2xy' - 2ny = 0$  in a special class of analytic functions.

Li and Shen [8] proved that if the characteristic equation  $\lambda^2 + \alpha\lambda + \beta = 0$  has two different positive roots, then the linear differential equation of second order with constant

coefficients  $y''(x) + \alpha y'(x) + \beta y(x) = f(x)$  has the Hyers-Ulam stability where  $y \in C^2[a, b]$ ,  $f \in C[a, b]$  and  $-\infty < a < b < +\infty$  (see also [9, 10]). Abdollahpour and Najati [11] proved that the third-order differential equation  $y^{(3)}(t) + \alpha y''(t) + \beta y'(t) + \gamma y(t) = f(t)$  has the Hyers-Ulam stability. Ghaemi *et al.* [12] proved the Hyers-Ulam stability of the exact second-order linear differential equation

$$p_0(x)y'' + p_1(x)y' + p_2(x)y + f(x) = 0$$

with  $p_0''(x) - p_1'(x) + p_2(x) = 0$ . Here  $p_0, p_1, p_2, f : (a, b) \rightarrow \mathbb{R}$  are continuous functions. For more results about the Hyers-Ulam stability of differential equations, we can refer to [13–21].

**Definition 1.2** We say that the differential equation

$$f(t)y''(t) + f_1(t)y'(t) + f_2(t)y(t) = Q(t), \quad (1.2)$$

is perfect if it can be written as  $\frac{d}{dt}[f(t)y'(t) + (f_1(t) - f'(t))y(t)] = Q(t)$ .

It is clear that the differential equation (1.2) is perfect if and only if  $f_2(t) = f_1'(t) - f''(t)$ . The aim of this paper is to investigate the Hyers-Ulam stability of the perfect differential equation (1.2), where  $f, y \in C^2[a, b]$ ,  $Q \in C[a, b]$ ,  $f_1 \in C^1[a, b]$ ,  $f_2(t) = f_1'(t) - f''(t)$  and  $-\infty < a < b < +\infty$ . More precisely, we prove that the equation (1.2) has the Hyers-Ulam stability.

## 2 Hyers-Ulam stability of the perfect differential equation

$$f(t)y''(t) + f_1(t)y'(t) + f_2(t)y(t) = Q(t)$$

In the following theorem, we prove the Hyers-Ulam stability of the differential equation (1.2).

Throughout this section,  $a$  and  $b$  are real numbers with  $-\infty < a < b < +\infty$ .

**Theorem 2.1** *The perfect differential equation*

$$f(t)y''(t) + f_1(t)y'(t) + f_2(t)y(t) = Q(t)$$

has the Hyers-Ulam stability, where  $f, y \in C^2[a, b]$ ,  $f_1 \in C^1[a, b]$ ,  $Q \in C[a, b]$  and  $f(t) \neq 0$  for all  $t \in [a, b]$ .

*Proof* Let  $\varepsilon > 0$  and  $y \in C^2[a, b]$  with

$$|f(t)y''(t) + f_1(t)y'(t) + f_2(t)y(t) - Q(t)| \leq \varepsilon.$$

Let  $g(t) = f(t)y'(t) + (f_1(t) - f'(t))y(t)$  for all  $t \in [a, b]$ . It is clear that

$$|g'(t) - Q(t)| = |f(t)y''(t) + f_1(t)y'(t) + f_2(t)y(t) - Q(t)| \leq \varepsilon.$$

We define

$$z(x) = g(b) - \int_x^b Q(t) dt, \quad x \in [a, b].$$

Then

$$z'(x) = Q(x), \quad x \in [a, b]. \quad (2.1)$$

Also, we have

$$\begin{aligned} |z(x) - g(x)| &= \left| g(b) - g(x) - \int_x^b Q(t) dt \right| = \left| \int_x^b g'(t) dt - \int_x^b Q(t) dt \right| \\ &\leq \int_x^b |g'(t) - Q(t)| dt \leq \varepsilon(b-a) \end{aligned}$$

for all  $x \in [a, b]$ . Now we define

$$F(x) = \frac{1}{f(x)} \exp \left\{ \int_a^x \frac{f_1(t)}{f(t)} dt \right\}, \quad u(x) = \frac{y(b)F(b)}{F(x)} - \frac{1}{F(x)} \int_x^b \frac{z(t)F(t)}{f(t)} dt$$

for all  $x \in [a, b]$ . It is clear that  $u \in C^2[a, b]$  and

$$u'(x)F(x) + u(x)F'(x) = \frac{z(x)F(x)}{f(x)}, \quad F'(x) = \frac{f_1(x) - f'(x)}{f(x)}F(x).$$

Therefore,

$$f(x)u'(x) + [f_1(x) - f'(x)]u(x) = z(x), \quad x \in [a, b]. \quad (2.2)$$

Hence, (2.1) implies that

$$f(x)u''(x) + f_1(x)u'(x) + f_2(x)u(x) = Q(x), \quad x \in [a, b].$$

Also, we have

$$\begin{aligned} |y(x) - u(x)| &= \left| y(x) - \frac{y(b)F(b)}{F(x)} + \frac{1}{F(x)} \int_x^b \frac{z(t)F(t)}{f(t)} dt \right| \\ &= \frac{1}{|F(x)|} \left| y(x)F(x) - y(b)F(b) + \int_x^b \frac{z(t)F(t)}{f(t)} dt \right| \\ &= \frac{1}{|F(x)|} \left| \int_x^b \frac{z(t)F(t)}{f(t)} dt - \int_x^b [y(t)F(t)]' dt \right| \\ &= \frac{1}{|F(x)|} \left| \int_x^b \left( \frac{z(t)F(t)}{f(t)} - y'(t)F(t) - y(t)F'(t) \right) dt \right| \\ &= \frac{1}{|F(x)|} \left| \int_x^b F(t) \left( \frac{z(t)}{f(t)} - y'(t) - \frac{f_1(t) - f'(t)}{f(t)} y(t) \right) dt \right| \\ &\leq \frac{1}{|F(x)|} \int_x^b \left| \frac{F(t)}{f(t)} \right| |z(t) - y'(t)f(t) - [f_1(t) - f'(t)]y(t)| dt \\ &= \frac{1}{|F(x)|} \int_x^b \left| \frac{F(t)}{f(t)} \right| |z(t) - g(t)| dt \\ &\leq \varepsilon(b-a) \frac{1}{|F(x)|} \int_x^b \left| \frac{F(t)}{f(t)} \right| dt \end{aligned} \quad (2.3)$$

for all  $x \in [a, b]$ . Since  $\frac{f_1}{f} \in C[a, b]$ , there exist constants  $m'$  and  $M'$  such that  $m' \leq \frac{f_1(x)}{f(x)} \leq M'$ . Thus

$$\begin{cases} 1 \leq \exp\{\int_a^x \frac{f_1(t)}{f(t)} dt\} \leq e^{M'(b-a)} & \text{if } m' \geq 0; \\ e^{m'(b-a)} \leq \exp\{\int_a^x \frac{f_1(t)}{f(t)} dt\} \leq e^{M'(b-a)} & \text{if } m' < 0 \leq M'; \\ e^{m'(b-a)} \leq \exp\{\int_a^x \frac{f_1(t)}{f(t)} dt\} \leq 1 & \text{if } M' < 0 \end{cases} \quad (2.4)$$

for all  $x \in [a, b]$ . Since  $f \in C[a, b]$  and  $|f| > 0$ , there exist constants  $0 < m \leq M$  such that  $m \leq |f(x)| \leq M$  for all  $x \in [a, b]$ . Hence, (2.4) implies that

$$\frac{1}{M} e^{m'|(a-b)|} \leq |F(x)| \leq \frac{1}{m} e^{M'|(b-a)|}$$

for all  $x \in [a, b]$ . It follows from (2.3) that

$$\begin{aligned} |y(x) - u(x)| &\leq \varepsilon(b-a) \frac{1}{|F(x)|} \int_a^b \left| \frac{F(t)}{f(t)} \right| dt \\ &\leq \varepsilon(b-a)^2 \frac{M}{m^2} e^{(|m'|+|M'|)(b-a)} \end{aligned}$$

for all  $x \in [a, b]$ . □

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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